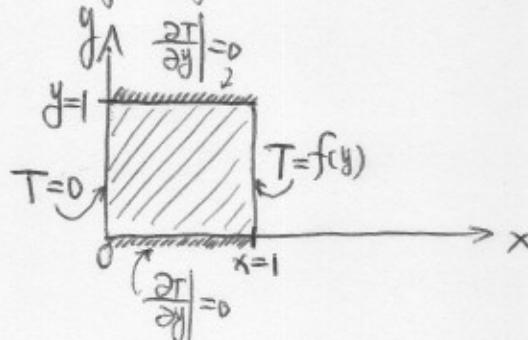


2.2. Separation of Variables — Rectangular System

* Idea — convert the given partial differential equation into multiple ordinary differential equations.

* Example 1

Consider a two-dimensional, steady-state heat conduction problem, without heat generation. The boundary conditions are specified in the schematic:



The complete problem (equation + boundary conditions).

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

B.C.

$$\left\{ \begin{array}{l} T|_{x=0} = 0 \\ T|_{x=1} = f(y) \\ (\frac{\partial T}{\partial y})|_{y=0} = 0 \\ (\frac{\partial T}{\partial y})|_{y=1} = 0 \end{array} \right.$$

← homogeneous

← homogeneous

← Nonhomogeneous !

← homogeneous

← homogeneous

To solve the problem,

assume $T(x,y) = X(x)Y(y)$ as a tentative solution,

$$\begin{cases} X(x) \text{ is a function of } x \text{ alone} \\ Y(y) \text{ is a function of } y \text{ alone} \end{cases}$$

Substitute $T(x,y) = X(x)Y(y)$ into original equation,

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

divide by the product $X(x)Y(y)$,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

i.e.,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

↓ ↓
 function of x only function of y only

Each side must be a constant not depending on x or y :

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \text{ (const.)}$$

Therefore, the determination of the solutions of the original PDE is reduced to solving two ODEs:

$\int X''(x) - \mu X(x) = 0 \quad (a)$	
$\int Y''(y) + \mu Y(y) = 0 \quad (b)$	

To solve the two reduced ODEs with μ as parameter,
there are three cases to investigate:

$$\underline{\mu < 0, \mu = 0, \mu > 0}$$

(1) $\mu < 0$

$$\text{Let } \mu = -\lambda^2 (\lambda > 0)$$

$$\begin{aligned} \text{We have: } & \left\{ \begin{array}{l} (a) \quad X''(x) + \lambda^2 X(x) = 0 \\ (b) \quad Y''(y) - \lambda^2 Y(y) = 0 \end{array} \right. \\ & \left. \begin{array}{l} X(x) = A \cos \lambda x + B \sin \lambda x \quad (\text{general solution}) \\ Y(y) = C \cosh \lambda y + D \sinh \lambda y \quad (\text{general solution}) \end{array} \right. \end{aligned}$$

Therefore, the general solution for $T(x, y)$ is:

$$\underline{T(x, y) = (A \cos \lambda x + B \sin \lambda x)(C \cosh \lambda y + D \sinh \lambda y)}$$

$$\underline{\text{Imposing B.C. } T|_{x=0} = 0}$$

$$T|_{x=0} = A(C \cosh \lambda y + D \sinh \lambda y) = 0 \quad (\text{for all } y)$$

There are two possibilities:

$$C=D=0 \quad \text{or} \quad A=0.$$

$$\text{If } C=D=0 \Rightarrow T(x, y)=0$$

not a meaningful solution!

So we must have:

$$\underline{\underline{A=0}}$$

$$\text{Therefore, } T(x, y) = B \sin \lambda x (C \cosh \lambda y + D \sinh \lambda y)$$

$$\text{i.e., } \underbrace{T(x, y)}_{\text{ }} = \sin \lambda x (C \cosh \lambda y + D \sinh \lambda y)$$

("B" can be absorbed into "C" and "D")

$$\underline{\text{Imposing B.C.}} \quad \frac{\partial T}{\partial y} \Big|_{y=0} = 0$$

$$\begin{aligned} \frac{\partial T}{\partial y} \Big|_{y=0} &= \sin \lambda x (C \lambda \sinh \lambda y + D \lambda \cosh \lambda y) \Big|_{y=0} \\ &= \sin \lambda x (D \lambda) = 0 \end{aligned}$$

$$\text{i.e., } D \lambda \sin \lambda x = 0 \quad (\text{for all } x)$$

$$\text{so: } \underline{D = 0}$$

$$\text{Therefore, } \underbrace{T(x, y)}_{\text{ }} = C \sin \lambda x \cosh \lambda y$$

$$\underline{\text{Imposing B.C.}} \quad \frac{\partial T}{\partial y} \Big|_{y=1} = 0$$

$$\begin{aligned} \frac{\partial T}{\partial y} \Big|_{y=1} &= C \sin \lambda x (\lambda \sinh \lambda y) \Big|_{y=1} \\ &= C \lambda \sinh \lambda \sin \lambda x = 0 \quad (\text{for all } x) \end{aligned}$$

$$\text{so: } \underline{C = 0}$$

$$\text{Therefore, } \underbrace{T(x, y)}_{\text{ }} = 0$$

It is not a meaningful solution!

Conclusion: μ cannot be smaller than 0. ~~($\lambda < 0$)~~

(2) $\mu = 0$

We have:

$$\left\{ \begin{array}{l} (a) \quad X''(x) = 0 \\ \qquad \qquad X(x) = Ax + B \quad (\text{general solution}) \\ (b) \quad Y''(y) = 0 \\ \qquad \qquad Y(y) = Cy + D \quad (\text{general solution}) \end{array} \right.$$

Therefore, the general solution for $T(x, y)$ is:

$$\underline{T(x, y) = (Ax + B)(Cy + D)}$$

Imposing B.C $T|_{x=0} = 0$

$$T|_{x=0} = B(Cy + D) = 0 \quad (\text{for all } y)$$

There are two possibilities:

$$C=D=0 \quad \text{or} \quad B=0$$

If $C=D=0 \Rightarrow T(x, y)=0$

not a meaningful solution!
So we must have:

$$\underline{\underline{B=0}}$$

Therefore, $T(x, y) = Ax(Cy + D)$

i.e., $\underline{T(x, y) = x(Cy + D)}$

Imposing B.C. $\frac{\partial T}{\partial y}|_{y=0} = 0$ ("A" can be absorbed into "C" and "D")

$$\frac{\partial T}{\partial y}|_{y=0} = x \cdot C = 0 \quad (\text{for all } x)$$

so: $\underline{\underline{C=0}}$

Therefore, $\underline{T(x,y) = Dx}$ (not a function of y)

naturally satisfy BC: $\frac{\partial T}{\partial y}|_{y=1} = 0$.

(3) $\mu > 0$

Let $\mu = \lambda^2$ ($\lambda > 0$)

We have:
$$\begin{cases} (a) \quad \ddot{X}(x) - \lambda^2 x = 0 \\ (b) \quad \dot{Y}'(y) + \lambda^2 y = 0 \end{cases}$$

$X(x) = A \cosh \lambda x + B \sinh \lambda x$ (general solution)

$Y(y) = C \cos \lambda y + D \sin \lambda y$ (general solution)

Therefore, the general solution for $T(x,y)$ is:

$$\underline{T(x,y) = (A \cosh \lambda x + B \sinh \lambda x)(C \cos \lambda y + D \sin \lambda y)}$$

Imposing B.C. $T|_{x=0} = 0$

$$T|_{x=0} = A(C \cos \lambda y + D \sin \lambda y) = 0 \quad (\text{for all } y)$$

There are two possibilities:

$$C=D=0 \quad \text{or} \quad A=0$$

If $C=D=0 \Rightarrow T(x,y)=0$

So we must have: $\underline{A=0}$ not a meaningful solution!

$$\underline{A=0}$$

Therefore, $\underline{T(x,y) = B \sinh \lambda x (C \cos \lambda y + D \sin \lambda y)}$

i.e., $\underline{T(x,y) = \sinh \lambda x (C \cos \lambda y + D \sin \lambda y)}$

("B" can be absorbed into "C" and "D")

Imposing B.C. $\left. \frac{\partial T}{\partial y} \right|_{y=0} = 0$

$$\begin{aligned}\left. \frac{\partial T}{\partial y} \right|_{y=0} &= \sinh \lambda x (-C \lambda \sin \lambda y + D \lambda \cos \lambda y) \Big|_{y=0} \\ &= \sinh \lambda x \cdot (D \lambda) = 0\end{aligned}$$

i.e. $D \lambda \sinh \lambda x = 0$ (for all x)

so: $D=0$

Therefore, $T(x, y) = C \sinh \lambda x \cos \lambda y$

Imposing B.C. $\left. \frac{\partial T}{\partial y} \right|_{y=1} = 0$

$$\begin{aligned}\left. \frac{\partial T}{\partial y} \right|_{y=1} &= C \sinh \lambda x (-\lambda \sin \lambda y) \Big|_{y=1} \\ &= C \sinh \lambda x (-\lambda \sin \lambda) = 0 \quad (\text{for all } x)\end{aligned}$$

so: $C \lambda \sin \lambda = 0$

There are two possibilities:

$$C=0 \quad \text{or} \quad \sin \lambda = 0 \quad (\text{Note: } \lambda > 0)$$

If $C=0 \Rightarrow T(x, y)=0$

not a meaningful solution!

Therefore, we must have:

$\sin \lambda = 0$

To satisfy this requirement, λ can only be discrete/special values:

$\lambda_n = n\pi$, $n=1, 2, 3, \dots$

(eigenvalues)

Therefore, for each n ($n=1, 2, 3, \dots$),

$$T_n(x, y) = C_n \sinh(\lambda_n x) \cos(n\pi y)$$

i.e., $\underbrace{T_n(x, y)}_{\text{(eigenfunction)}} = C_n \sinh(n\pi x) \cos(n\pi y)$

Note: $T_n(x, y)$ are the solutions of the original equation that satisfy three (of four) boundary conditions

$$\begin{cases} T|_{x=0} = 0 & \leftarrow \text{homogeneous} \\ \frac{\partial T}{\partial y}|_{y=0} = 0 & \leftarrow \text{homogeneous} \\ \frac{\partial T}{\partial y}|_{y=1} = 0 & \leftarrow \text{homogeneous} \end{cases}$$

In general, the forth boundary condition (nonhomogeneous), $T|_{x=1} = f(y)$, cannot be met by any one of the $T_n(x, y)$.

The final solution:

Because the equation is linear, the sums of solutions are also solutions, therefore, including all possible solutions,

$$T(x, y) = \underbrace{Dx}_{\substack{\uparrow \\ (\text{from } \mu=0)}} + \underbrace{\sum_{n=1}^{\infty} C_n \sinh(n\pi x) \cos(n\pi y)}_{\substack{\uparrow \\ (\text{from } \mu>0)}}$$

$$\text{Let } D = \frac{1}{2} C_0$$

Therefore,

$$T(x, y) = \frac{1}{2} C_0 x + \sum_{n=1}^{\infty} C_n \sinh(n\pi x) \cos(n\pi y)$$

Linear superposition!

(general solution.)

Applying BC. $T|_{x=1} = f(y) \leftarrow \text{Nonhomogeneous!}$

$$T|_{x=1} = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} [C_n \sinh(n\pi)] \cos(n\pi y) = f(y)$$

The unknown coefficients C_0 and C_n are determined by making use of the orthogonal property of the eigenfunctions.

$$\begin{cases} \int_0^1 \frac{1}{2}C_0 dy = \int_0^1 f(y) dy \\ \int_0^1 [C_n \sinh(n\pi y)] \cos(n\pi y) \cdot \cos(n\pi y) dy = \int_0^1 f(y) \cos(n\pi y) dy \end{cases}$$

i.e.
$$\boxed{\begin{cases} C_0 = 2 \int_0^1 f(y) dy \\ C_n = \frac{2}{\sinh(n\pi)} \int_0^1 f(y) \cos(n\pi y) dy \end{cases}}$$

With C_0 and C_n determined, the solution is complete!